

Potentials of the Heun class: the triconfluent case

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(Dated: December 15, 2014)

Since the advent of quantum mechanics different approaches to find analytical solutions of the Schrödinger equation have been successfully developed. Here we follow and generalize the approach pioneered by Natanzon and others by which the Schrödinger equations can be transformed into another well-known equation for transcendental function (e.g., the hypergeometric equation). This sets a class of potentials for which this transformation is possible. Our generalization consists in finding potentials allowing the transformation of the Schrödinger equation into a triconfluent Heun equation. We find the energy eigenvalues of this class of potentials, the eigenfunction and the exact superpartners.

PACS numbers: XXX

I. INTRODUCTION

The study of exactly solvable Schrödinger equations can be traced back to the beginning of quantum mechanics. The earliest examples are represented by the harmonic oscillator, Coulomb, Morse, Pöschl-Teller, Eckart and Manning-Rosen potentials [1–3]. By looking closer at these examples one can start to conjecture that exact solvability of the Schrödinger equation depends on the fact that such an equation can be suitably reduced to the hypergeometric or the confluent hypergeometric equation. The problem of deriving the most general class of potentials such that the Schrödinger equation can be transformed into the hypergeometric equation has been solved by [4]. Furthermore, [5] studied solutions regular at infinity for the basic SUSY ladder of Hyperbolic Pöschl-Teller potentials that admit representations in terms of confluent Heun polynomials. [6] derived new classes of potentials such that the one-dimensional Schrödinger equation can be turned into the Heun equation and its confluent cases. The generalized Heun equation has been considered as well in [6]. Since the hypergeometric equation

$$y(1-y)\frac{d^2v}{dy^2} + [c - (a+b+1)y]\frac{dv}{dy} - abv(y) = 0, \quad y \in I \subset \mathbb{R} \quad (1)$$

with $a, b, c \in \mathbb{R}$ is a special case of the Heun equation

$$\frac{d^2v}{dy^2} + \left(\frac{\gamma}{y} + \frac{\delta}{y-1} + \frac{\epsilon}{y-a} \right) \frac{dv}{dy} + \frac{\alpha\beta y - q}{y(y-1)(y-a)} v(y) = 0, \quad y \in I \subset \mathbb{R}, \quad (2)$$

whenever $q = \alpha\beta a$ and $\epsilon = 0$, the potential classes obtained in [6] generalizes the Natanzon's class. We start by reviewing the method developed by [6, 7] allowing the construction of the most general potential such that the Schrödinger equation can be reduced to the triconfluent Heun equation (5). Such a potential contains six free parameters.

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II. REDUCTION OF THE RADIAL SCHRÖDINGER EQUATION TO A TRICONFLUENT HEUN EQUATION

We consider a quantum particle in a central field in three and two spatial dimensions, respectively. In the three dimensional case the behaviour of the particle is described by the Schrödinger equation ($\hbar^2 = 2m = 1$)

$$i\partial_t\Psi(t, \mathbf{r}) = H\Psi(t, \mathbf{r}), \quad H = -\Delta + U(r)$$

with $\Psi \in L^2(\mathbb{R}^3)$. Here, Δ denotes the Laplacian. Since the Hamilton operator commutes with the angular momentum operator $\mathbf{L} = (L_x, L_y, L_z)$, it is sufficient to solve the eigenvalue problem for H , i.e. the time-dependent Schrödinger equation, on the subspace of $L^2(\mathbb{R}^3)$ spanned by a basis of common eigenvectors of $|\mathbf{L}|^2$ and L_z . Such a subspace is made of functions having the form $\Phi_{\ell m}(r, \theta, \phi) = \Phi(r)Y_{\ell m}(\theta, \phi)$ where $\Phi \in L^2(\mathbb{R}_+, r^2 dr)$ and $Y_{\ell m}(\theta, \phi)$ denotes the spherical harmonics. In spherical coordinates the Laplace operator becomes

$$\Delta = \frac{1}{r^2}\partial_r r^2 \partial_r - \frac{|\mathbf{L}|^2}{r^2}.$$

If $\Phi_{\ell m}$ is an eigenfunction of H relative to the eigenvalue E , then Φ must satisfy the radial Schrödinger equation [8]

$$\left[-\left(\frac{d^2}{dr^2} + \frac{2}{r} \frac{d}{dr} - \frac{\ell(\ell+1)}{r^2} \right) + U(r) \right] \Phi(r) = E\Phi(r).$$

The above equation can be further simplified if we bring it to its canonical form by means of the transformation $\Phi(r) = \psi(r)/r$ with $\psi \in L^2(\mathbb{R}_+, r^2 dr)$ and we obtain

$$\frac{d^2\psi}{dr^2} + [E - V_{eff}(r)]\psi(r) = 0, \quad V_{eff}(r) = \frac{\ell(\ell+1)}{r^2} + U(r). \quad (3)$$

The region where the dynamics of the particle can take place is constrained to the half axis $r > 0$. Moreover, it will be assumed that the effective potential does not depend on the energy of the particle. Two-dimensional quantum systems appear in solid state physics in connection with the fractional quantum Hall effect [9] and high temperature superconductivity [10–12]. For the two-dimensional case we consider a system of two anyons [13], i.e. particles with fractional statistics, in a spherically-symmetric potential. The time-independent Schrödinger equation governing this system is [14, 15] ($\hbar^2 = 2\mu = 1$ with μ the reduced mass of the system)

$$\left[-\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{\nu^2}{r^2} \right) + U(r) \right] \Phi(r) = E\Phi(r), \quad \nu = m + \frac{\theta}{\pi},$$

where θ is the statistics parameter, $m \in \mathbb{Z}$ is the azimuthal quantum number and $r = |\mathbf{r}_1 - \mathbf{r}_2| > 0$ is the relative coordinate. Note that $r \neq 0$ to avoid that the two particles overlap. By means of the substitution $\Phi(r) = \psi(r)/\sqrt{r}$ we get formally the same Schrödinger equation as given by (3) with

$$V_{eff}(r) = \frac{\nu^2 - \frac{1}{4}}{r^2} + U(r). \quad (4)$$

We want to construct the most general potential U such that the radial Schrödinger equation (3) with effective potential given by (3) with $\ell = 0$ or (4) with fixed ν can be transformed into the triconfluent Heun equation [16]

$$\frac{d^2v}{d\rho^2} + I(\rho)v(\rho) = 0, \quad I(\rho) = A_0 + A_1\rho + A_2\rho^2 - \frac{9}{4}\rho^4 \quad (5)$$

with parameters $A_1, A_2, A_2 \in \mathbb{R}$ and $\rho \in \Omega \subseteq \mathbb{R}$. Note that (5) can be obtained by means of a confluence process of the singularities involved in the Heun equation. To appreciate the role of the TCH equation in physics, we refer to [17–20] where this equation appears in the treatment of anharmonic oscillators in quantum mechanics. A short exposition of known results concerning anharmonic oscillators can be found in Ch. I of [21]. It is interesting to observe that the radial Schrödinger equation (3) reduces immediately to the above triconfluent Heun equation whenever

$$r = \rho, \quad \psi = v, \quad E = A_0, \quad \ell = 0, \quad U(r) = -A_1r - A_2r^2 + \frac{9}{4}r^4,$$

i.e. U is a quartic potential. In what follows we are interested in non trivial transformations of the dependent variable ψ and the radial coordinate r transforming (3) into (5). As in [6] we introduce the coordinate transformation $\rho = \rho(r)$ with $r > 0$ and (5) becomes

$$\frac{d^2 v}{dr^2} - \frac{\rho''}{\rho'} \frac{dv}{dr} + (\rho')^2 I(\rho(r)) v(\rho(r)) = 0, \quad ' := \frac{d}{dr}.$$

The standard form of the above equation can be achieved by using the Liouville transformation

$$v(\rho(r)) = \exp\left(\frac{1}{2} \int \frac{\rho''(r)}{\rho'(r)} dr\right) \psi(r) = \sqrt{\rho'(r)} \psi(r),$$

where we must require that

$$\rho'(r) > 0 \tag{6}$$

on the interval $(0, \infty)$, i.e. ρ must be an increasing function of the variable r . Hence, we end up with the linear ODE

$$\frac{d^2 \psi}{dr^2} + J(r) \psi(r) = 0, \quad J(r) = (\rho')^2 I(\rho(r)) + \frac{1}{2} S(\rho). \tag{7}$$

Here, $S(\rho)$ denotes the Schwarzian derivative of the coordinate transformation ρ evaluated at r and it is given by

$$S(\rho) = \frac{\rho'''}{\rho'} - \frac{3}{2} \left(\frac{\rho''}{\rho'} \right)^2.$$

It can be easily checked that solutions of (7) will be expressed in terms of the solutions of (5) according to

$$\psi(r) = \frac{1}{\sqrt{\rho'(r)}} v(\rho(r)).$$

Moreover, equation (7) will reduce to the radial Schrödinger equation (3) when $J(r) = E - V_{eff}(r)$. Therefore, the effective potential is completely determined by the Bose invariant I [22] and the Schwarzian derivative of the coordinate transformation. In order to be sure that the potential does not depend on the energy of the particle, we must require that

- the Bose invariant admits a decomposition of the form

$$I(\rho) = I_0(\rho) + E I_1(\rho). \tag{8}$$

According to Theorem IV.6 in [6], this will be the case if the parameters entering in the triconfluent Heun equation can be written as $A_i = a_i + E b_i$ with $i = 0, 1, 2$ and $a_i, b_i \in \mathbb{R}$ for any $i = 0, 1, 2$. Then, we have

$$I_0(\rho) = a_0 + a_1 \rho + a_2 \rho^2 - \frac{9}{4} \rho^4, \quad I_1(\rho) = b_0 + b_1 \rho + b_2 \rho^2.$$

- The coordinate transformation satisfies the differential equation

$$(\rho')^2 I_1(\rho(r)) = 1. \tag{9}$$

The condition expressed by (6) implies that we have to take the positive square root of (9), that is

$$\rho'(r) = \frac{1}{\sqrt{I_1(\rho(r))}}. \tag{10}$$

With the help of (8) and (9) the effective potential can be written as

$$V_{eff}(r) = -\frac{I_0(\rho(r))}{I_1(\rho(r))} - \frac{1}{2} S(\rho).$$

Rewriting the Schwarzian derivative in terms of I_1 and its derivatives with respect to ρ we obtain the most general form for the effective potential such that the radial Schrödinger equation (3) can be transformed into a triconfluent Heun equation, namely

$$V_{eff}(r) = -\frac{I_0(\rho(r))}{I_1(\rho(r))} + \frac{4I_1(\rho(r))\ddot{I}_1(\rho(r)) - 5\dot{I}_1^2(\rho(r))}{16I_1^3(\rho(r))}, \quad \cdot := \frac{d}{d\rho}. \quad (11)$$

Replacing the corresponding expressions for I_0 and I_1 into (11) we can write the effective potential as

$$V_{eff}(r) = -\frac{4a_0 + 4a_1\rho + 4a_2\rho^2 - 9\rho^4}{4(b_0 + b_1\rho + b_2\rho^2)} - \frac{12b_2^2\rho^2 + 12b_1b_2\rho + 5b_1^2 - 8b_0b_2}{16(b_0 + b_1\rho + b_2\rho^2)^3}, \quad \rho = \rho(r). \quad (12)$$

Note that (12) represents a class of potentials depending upon the six real parameters a_i, b_i with $i = 0, 1, 2$. Concerning the solution of the differential equation governing the coordinate transformation we need to require that $I_1(\rho(r)) > 0$ for $r \in (0, \infty)$. Let $\Delta = b_1^2 - 4b_0b_2$. We have the following cases

1. $b_0, b_1, b_2 \neq 0$ and $b_2 > 0$.

(a) If $\Delta < 0$, the coordinate transformation ρ maps the interval $(0, \infty)$ of the radial variable into the whole real line. Using 2.261 and 2.262(1) in [23] we find that the solution of (10) can be written as

$$r(\rho) = \frac{1}{2} \left(\rho + \frac{b_1}{2b_2} \right) \sqrt{I_1} - \frac{\Delta}{8b_2\sqrt{b_2}} \operatorname{arcsinh} \left(\frac{2b_2\rho + b_1}{\sqrt{-\Delta}} \right).$$

As $\rho \rightarrow \infty$ we see that $r \approx (\sqrt{b_2}/2)\rho^2$.

(b) Let $\Delta = 0$. Since $b_1^2 > 0$, $b_2 > 0$, and $b_1^2 = 4b_0b_2$, then $b_0 > 0$ and $b_1 = \pm 2\sqrt{b_0b_2}$. Hence, $I_1 = (\sqrt{b_2}\rho \pm \sqrt{b_0})^2$ and equation (10) becomes

$$\rho'(r) = \frac{1}{\sqrt{b_2}\rho \pm \sqrt{b_0}},$$

where the plus sign must be taken when $b_1 > 0$. Note that $\rho' > 0$ whenever $\rho > \mp \sqrt{b_0/b_2}$ where the minus sign must be taken for $b_1 > 0$. Integrating the above equation we obtain

$$r(\rho) = \frac{\sqrt{b_2}}{2}\rho^2 \pm \sqrt{b_0}\rho.$$

The above equation is quadratic in ρ and therefore we can solve it in order to express ρ as a function of the radial variable. In the case $b_1 > 0$ and requiring that ρ is an increasing function of the radial variable we find that

$$\rho(r) = \frac{-\sqrt{b_0} + \sqrt{b_0 + 2\sqrt{b_2}r}}{\sqrt{b_2}}.$$

In this case ρ maps the interval $(0, \infty)$ into itself and the condition $\rho > -\sqrt{b_0/b_2}$ is automatically satisfied. If $b_1 < 0$, by a similar reasoning we find the solution

$$\rho(r) = \frac{\sqrt{b_0} + \sqrt{b_0 + 2\sqrt{b_2}r}}{\sqrt{b_2}}.$$

The image of the interval $(0, \infty)$ under the transformation ρ is the interval $(2\sqrt{b_0/b_2}, \infty)$ and also in this case the condition $\rho > \sqrt{b_0/b_2}$ is satisfied. If $b_0 = 0$, then $\Delta = 0$ implies $b_1 = 0$ and ρ is given by the expression

$$\rho(r) = \sqrt{\frac{2r}{\sqrt{b_2}}}.$$

(c) If $\Delta > 0$, then $I_1 > 0$ on the interval $(-\infty, \rho_1) \cup (\rho_2, \infty)$ where ρ_1 and ρ_2 denote the roots of I_1 . In this case we can express the radial variable in terms of ρ as

$$r(\rho) = \frac{1}{2} \left(\rho + \frac{b_1}{2b_2} \right) \sqrt{I_1} - \frac{\Delta}{8b_2\sqrt{b_2}} \ln \left(2\sqrt{b_2I_1} + 2b_2\rho + b_1 \right).$$

2. $b_0, b_1, b_2 \neq 0$ and $b_2 < 0$. In this case only if $\Delta > 0$ we can make $I_1 > 0$ on an open interval (ρ_1, ρ_2) where ρ_1 and ρ_2 denote the roots of I_1 . Employing 2.261 and 2.262(1) in [23] yield

$$r(\rho) = \frac{1}{2} \left(\rho + \frac{b_1}{2b_2} \right) \sqrt{I_1} + \frac{\Delta}{8b_2\sqrt{-b_2}} \arcsin \left(\frac{2b_2\rho + b_1}{\sqrt{\Delta}} \right).$$

3. If $b_2 = 0$ and $b_1 \neq 0$, then $I_1(\rho) = b_1\rho + b_0$ and we must require that $\rho > -b_0/b_1$. The solution of the ODE governing the coordinate transformation is

$$r(\rho) = \frac{2}{3b_1} (b_1\rho + b_0)^{3/2}.$$

Moreover, we can express ρ in terms of the radial coordinate as

$$\rho(r) = -\frac{b_0}{b_1} + \frac{1}{b_1} \left(\frac{3b_1r}{2} \right)^{2/3}.$$

We can observe that ρ is an increasing function of r and it maps the interval $(0, \infty)$ to $(-b_0/b_1, \infty)$. If $b_0 = 0$ the above expression simplifies to

$$\rho(r) = \left(\frac{3}{2} \right)^{2/3} b_1^{-1/3} r^{2/3}.$$

III. ANALYSIS OF THE POTENTIALS

We analyze those potentials arising from the cases when the coordinate transformation ρ can be written as an explicit function of the real variable r .

A. The case $\Delta = 0$, and $b_0, b_1, b_2 > 0$

It is straightforward to verify that $I_1 = b_0 + 2\sqrt{b_2}r$ and the effective potential can be written as

$$V_1(r) = c_0 + \frac{c_1}{\sqrt{b_0 + 2\sqrt{b_2}r}} + \frac{c_2}{b_0 + 2\sqrt{b_2}r} + \frac{c_3}{(b_0 + 2\sqrt{b_2}r)^2} + c_4\sqrt{b_0 + 2\sqrt{b_2}r} + c_5r \quad (13)$$

with

$$c_0 = \frac{63b_0}{4b_2^2} - \frac{a_2}{b_2}, \quad c_1 = \frac{2a_2\sqrt{b_0}}{b_2} - \frac{a_1}{\sqrt{b_2}} - \frac{9b_0\sqrt{b_0}}{b_2^2}, \quad c_2 = -a_0 + \frac{a_1\sqrt{b_0}}{\sqrt{b_2}} + \frac{9b_0^2}{4b_2^2} - \frac{a_2b_0}{b_2},$$

$$c_3 = -\frac{3}{4}b_2, \quad c_4 = -\frac{9\sqrt{b_0}}{b_2^2}, \quad c_5 = \frac{9}{2b_2\sqrt{b_2}}$$

Note that $c_0 = c_1 = c_2 = 0$ if we choose

$$a_0 = 9 \left(\frac{b_0}{b_2} \right)^2, \quad a_1 = \frac{45}{2} \left(\frac{b_0}{b_2} \right)^{3/2}, \quad a_2 = \frac{63b_0}{4b_2}.$$

and the corresponding effective potential will depend only on the parameters b_0 and b_2 . We observe that

1. Since $r > 0$ and $b_0 > 0$, it follows that the denominators appearing in (13) can never vanish and therefore the potential is never singular on the open interval $(0, \infty)$.
2. If $r = 0$ the potential is finite there while asymptotically for $r \rightarrow \infty$ it grows linearly as

$$V_1(r) \approx \frac{9r}{2b_2\sqrt{b_2}}.$$

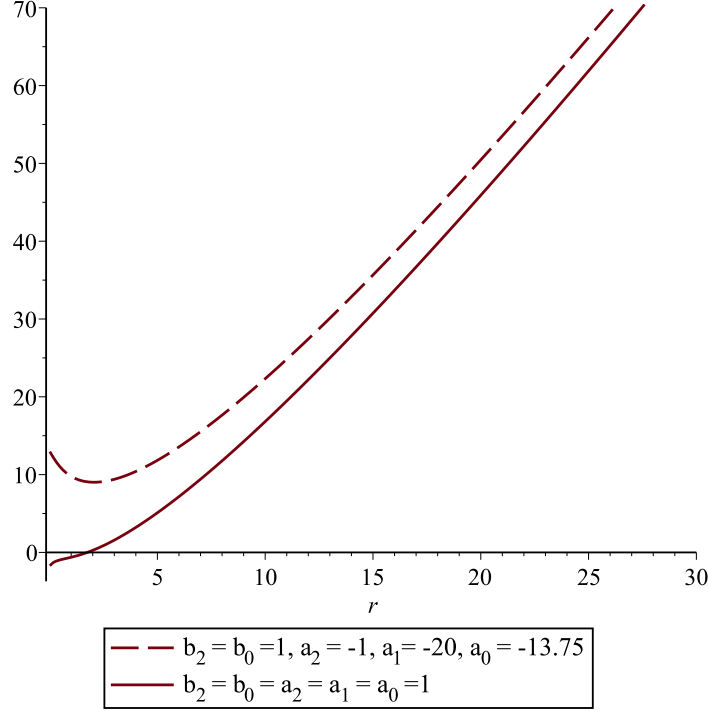


FIG. 1: Behaviour of the potential (13) for different choices of the parameters.

3. There are choices of the parameters such that V_1 has at least a global minimum on the positive real line. See Fig. 1.

The study of the critical points of this potential leads to a polynomial equation of degree eight. Applying Descartes' rule of signs we find that the potential (13) has four, two, or no positive critical points whenever c_1 and c_3 are negative and c_2 is positive; three or one positive critical point if $c_1 < 0$ and $c_2, c_3 > 0$, and only one positive critical point if c_1, c_2 , and c_3 are positive. For other choices of the signs of the coefficients c_1, c_2 , and c_3 there are two or no positive critical point.

B. The case $\Delta = 0$, $b_0, b_2 > 0$, and $b_1 < 0$

The function I_1 is formally given as in the previous case and the effective potential has the same form as (13) with coefficients d_0, \dots, d_5 given by

$$d_0 = c_0, \quad d_1 = -\frac{2a_2\sqrt{b_0}}{b_2} - \frac{a_1}{\sqrt{b_2}} + \frac{9b_0\sqrt{b_0}}{b_2^2}, \quad d_2 = a_0 - \frac{a_1\sqrt{b_0}}{\sqrt{b_2}} + \frac{9b_0^2}{4b_2^2} - \frac{a_2b_0}{b_2}, \quad d_3 = c_3, \quad d_4 = -c_4, \quad d_5 = c_5.$$

Note that $d_0 = d_1 = d_2 = 0$ if we choose

$$a_0 = -36 \left(\frac{b_0}{b_2} \right)^2, \quad a_1 = -\frac{45}{2} \left(\frac{b_0}{b_2} \right)^{3/2}, \quad a_2 = \frac{63b_0}{4b_2}.$$

We observe that

1. Since $r > 0$ and $b_0 > 0$, it follows that I_1 can never vanish and therefore the potential is never singular.
2. The potential is finite at $r = 0$ while asymptotically for $r \rightarrow \infty$ it grows linearly as (13).
3. There are again choices of the parameters such that the potential has at least one minimum.

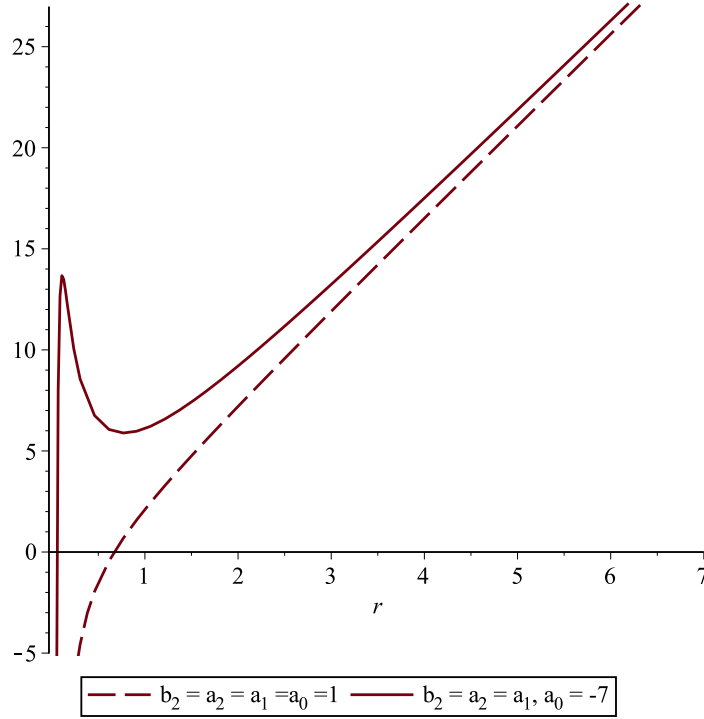


FIG. 2: Behaviour of the potential for the case $\Delta = 0$, $b_0, b_2 > 0$, and $b_1 < 0$.

C. The case $\Delta = 0$, $b_2 > 0$, and $b_0 = b_1 = 0$

The potential takes the form

$$V_3(r) = e_0 + \frac{e_1}{\sqrt{r}} + \frac{e_2}{r} + \frac{e_3}{r^2} + e_4 r \quad (14)$$

where

$$e_0 = -\frac{a_2}{b_2}, \quad e_1 = -\frac{a_1}{\sqrt{2}b_2^{3/4}}, \quad e_2 = -\frac{a_0}{2\sqrt{b_2}}, \quad e_3 = -\frac{3}{16}, \quad e_4 = c_5.$$

Note that it becomes singular at $r = 0$ and asymptotically at infinity it behaves as (13). There are choices of the parameters such that this potential has a maximum very close to $r = 0$ and a minimum. See Fig. 2. The potential (14) is singular at $r = 0$ and it grows linearly for $r \rightarrow \infty$. We have choices of the parameters such that the potential exhibits maxima and minima. If we make the substitution $r = \tau^2$, the critical points of the potential (14) must satisfy the sextic equation

$$2e_4\tau^6 - e_1\tau^3 - 2e_2\tau^2 - 4e_3 = 0. \quad (15)$$

If a_0 and a_1 are both positive, there is no change of sign in (15) and Descartes' rule of signs implies that the sextic equation will not have any positive real root, and hence the potential has no extrema. For all other choices of the signs of the coefficients a_0 , and a_1 , there are only two sign changes in (15) and hence the potential will admit two positive critical points or none. For a complete root classification of a sextic equation we refer to [24].

D. The case $b_2 = 0$

It can be easily checked that $I_1(r) = (3b_1r/2)^{2/3}$ and the potential can be written as

$$V_4(r) = v_0 + \frac{v_1}{r^{2/3}} + \frac{v_2}{r^2} + v_3r^{2/3} + v_4r^{4/3} + v_5r^2 \quad (16)$$

with

$$v_0 = -\frac{a_1}{b_1} + \frac{2a_2b_0}{b_1^2} - \frac{9b_0^3}{b_1^4}, \quad v_1 = \left(\frac{2}{3b_1}\right)^{2/3} \left(\frac{a_1b_0}{b_1} - a_0 - \frac{a_2b_0^2}{b_1^2} + \frac{9b_0^4}{4b_1^4}\right), \quad v_2 = -\frac{5}{36},$$

$$v_3 = \frac{1}{b_1^{2/3}} \left(\frac{2}{3b_1}\right)^{2/3} \left[\frac{27}{2} \left(\frac{b_0}{b_1}\right)^2 - a_2\right], \quad v_4 = -\frac{3^{10/3}b_0}{2^{4/3}b_1^{8/3}}, \quad v_5 = \frac{81}{16b_1^2}.$$

We can set $v_0 = v_1 = v_3 = 0$ if we choose

$$a_0 = \frac{27}{4} \left(\frac{b_0}{b_1}\right)^4, \quad a_1 = 18 \left(\frac{b_0}{b_1}\right)^3, \quad a_2 = \frac{27}{2} \left(\frac{b_0}{b_1}\right)^2.$$

The potential (16) is singular at $r = 0$ and it grows quadratically for $r \rightarrow \infty$. We have choices of the parameters such that the potential exhibits maxima and minima. If we make the substitution $r = \tau^{3/2}$, it is not difficult to verify that the critical points of the potential (16) must satisfy the sextic equation

$$3v_5\tau^6 + 2v_4\tau^5 + v_3\tau^4 - v_1\tau^2 - 3v_2 = 0. \quad (17)$$

If $v_1 < 0$ and $v_3, v_4 > 0$, we do not have any change of sign in (17) and Descartes' rule of signs implies that the sextic equation will not have any positive real root, and therefore the potential has no extrema. If $v_1, v_2 > 0$ and $v_4 < 0$, there are four changes of sign and the potential may have four, two or no positive critical point. For all other choices of the signs of the coefficients v_1, v_3 , and v_4 there are only two sign changes in (17) and hence the potential will admit two positive critical points or none. In the subcase $b_2 = b_0 = 0$ the potential (16) simplifies to

$$V_5(r) = u_0 + \frac{u_1}{r^{2/3}} + \frac{u_2}{r^2} + u_3r^{2/3} + u_4r^2 \quad (18)$$

where

$$u_0 = -\frac{a_1}{b_1}, \quad u_1 = -a_0\sqrt[3]{\frac{4}{9b_1^2}}, \quad u_2 = v_2, \quad u_3 = -a_2\sqrt[3]{\frac{4}{9b_1^4}}, \quad u_4 = \frac{81}{16b_1^2}.$$

If we make the substitution $r = \tau^{3/4}$, it is not difficult to verify that the critical points of the potential (18) must satisfy the cubic equation

$$3u_4\tau^3 + u_3\tau^2 - u_1\tau - 3u_2 = 0. \quad (19)$$

If $a_2 < 0$ and $a_0 > 0$, we do not have any change of sign in (19) and Descartes' rule of signs implies that the cubic equation will not have any positive real root. Hence, the potential has no positive extrema. For any other choice of the signs of the parameters a_2 and a_1 , we will always have two sign changes in (19) and therefore the potential will admit two positive critical points or none.

IV. SOLUTIONS OF THE RADIAL SCHRÖDINGER EQUATION

We study the behaviour of the solutions of the radial Schrödinger equation for the general class of potentials (12) and for the potentials discussed in the previous section. First of all, by means of (10) the solution of the radial Schrödinger equation can be written as

$$\psi(r) = \sqrt[4]{b_0 + b_1\rho + b_2\rho^2}v(\rho),$$

where $\rho = \rho(r)$ and v is a solution of the triconfluent Heun equation (5). It is in general convenient to bring (5) into its canonical form [16] with the help of the substitution

$$v(\rho) = e^{\frac{A_2}{3}\rho - \frac{\rho^3}{2}}y(\rho),$$

where the unknown function y must satisfy the equation

$$\frac{d^2y}{d\rho^2} - (\gamma + 3\rho^2)\frac{dy}{d\rho} + [\alpha + (\beta - 3)\rho] = 0 \quad (20)$$

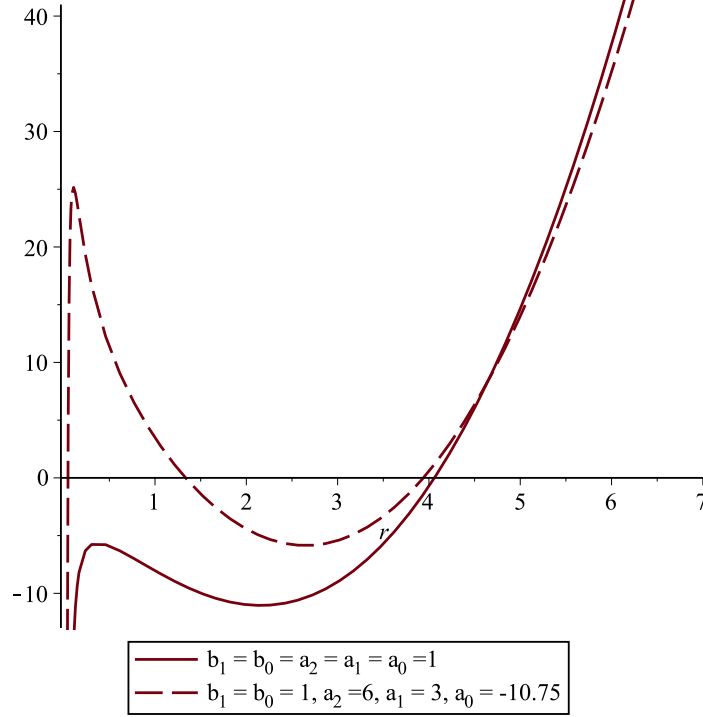


FIG. 3: Behaviour of the potential (16) for different values of the parameters.

with

$$\alpha = A_0 + \frac{A_2^2}{9}, \quad \beta = A_1, \quad \gamma = -\frac{2}{3}A_2. \quad (21)$$

Hence, the general solution of the radial Schrödinger equation can be cast in the form

$$\psi(r) = \sqrt[4]{b_0 + b_1\rho + b_2\rho^2} e^{\frac{A_2}{3}\rho - \frac{\rho^3}{2}} y(\rho). \quad (22)$$

Note that the function multiplying y in (22) decays exponentially in r as $r \rightarrow +\infty$ because in the case $b_0, b_1, b_2 \neq 0$ we have

$$\rho(r) \approx \begin{cases} \sqrt{\frac{2r}{\sqrt{b_2}}} & \text{if } b_2 > 0 \\ \rho_2 + \frac{1}{\sqrt[3]{|b_2|(\rho_2 - \rho_1)}} \left(\frac{3r}{2}\right)^{2/3} & \text{if } b_2 < 0. \end{cases}$$

with ρ_1 and ρ_2 denoting the roots of the quadratic polynomial I_1 introduced in Section II whereas if $b_2 = 0$ and $b_1 \neq 0$ we get

$$\rho(r) \approx \frac{1}{b_1} \left(\frac{3b_1 r}{2} \right)^{2/3}.$$

Since equation (5) has no finite singular points, we can construct Taylor series for the solutions of the triconfluent Heun equation by adopting the method outlined in [16]. We first rewrite (20) in terms of the Euler operator $\delta = \rho d/d\rho$ as

$$\delta^2 y - (1 + \gamma\rho + 3\rho^3)\delta y + (\beta - 3)\rho^3 y = 0.$$

Observing that ρ^n with $n \in \mathbb{N}$ are eigenfunctions of the Euler operator, one finds the following two linearly independent solutions, namely

$$T_1(\alpha, \beta, \gamma; \rho) = \sum_{n=0}^{\infty} e_n(\alpha, \beta, \gamma) \rho^n, \quad T_2(\alpha, \beta, \gamma; \rho) = \sum_{n=0}^{\infty} s_n(\alpha, \beta, \gamma) \rho^{n+1} \quad (23)$$

with

$$e_0(\alpha, \beta, \gamma) = 1, \quad e_1(\alpha, \beta, \gamma) = 0, \quad e_2(\alpha, \beta, \gamma) = -\frac{\alpha}{2},$$

$$e_n(\alpha, \beta, \gamma) = \frac{(n-1)\gamma e_{n-1}(\alpha, \beta, \gamma) - \alpha e_{n-2}(\alpha, \beta, \gamma) - (\beta + 6 - 3n)e_{n-3}(\alpha, \beta, \gamma)}{n(n-1)}, \quad n \geq 3$$

and

$$s_0(\alpha, \beta, \gamma) = 1, \quad s_1(\alpha, \beta, \gamma) = \frac{\gamma}{2}, \quad s_2(\alpha, \beta, \gamma) = \frac{\gamma^2 - \alpha}{6},$$

$$s_n(\alpha, \beta, \gamma) = \frac{n\gamma s_{n-1}(\alpha, \beta, \gamma) - \alpha s_{n-2}(\alpha, \beta, \gamma) - (\beta + 3 - 3n)s_{n-3}(\alpha, \beta, \gamma)}{n(n+1)}, \quad n \geq 3.$$

We will refer to (23) as the general solution to contrast it with the polynomial one which will be discussed below. Worth mentioning is also the convergence of (23) as shown in [16].

In order to find polynomial solutions of (20) we start AGAIN by assuming a power solution of the form

$$y(\rho) = \sum_{n=0}^{\infty} w_n \rho^n. \quad (24)$$

Substituting (24) into (20) we obtain the following recurrence relation for the coefficients w_n

$$(\beta - 3n)w_{n-1} + \alpha w_n - \gamma(n+1)w_{n+1} + (n+1)(n+2)w_{n+2} = 0 \quad (25)$$

which holds for all $n \geq 0$ provided that $w_{-1} = 0$. The above recursion relation is the same as the recursion relation for e_n (see above) with a shift for n . In order to have a polynomial solution of degree N we require that $w_n = 0$ for all $n > N$. Moreover, for $n = N+1$ the recurrence relation above gives the condition $\beta = 3(N+1)$. Taking into account that $\beta = a_1 + b_1 E$, such a condition gives the energy eigenvalue

$$E_N = \frac{3(N+1) - a_1}{b_1}, \quad b_1 \neq 0$$

with $N = 0, 1, 2, \dots$. Furthermore, for $0 \leq n \leq N$ the recursion formula (25) generates a system of $N+1$ homogenous linear equations for the coefficients w_0, w_1, \dots, w_N . This system admits a non-trivial solution if and only if the determinant of the $(N+1) \times (N+1)$ matrix

$$D_{N+1} = \begin{pmatrix} \alpha & -\gamma & 2 & 0 & 0 & \dots & 0 \\ 3N & \alpha & -2\gamma & 2 \cdot 3 & 0 & \dots & \\ 0 & 3(N-1) & \alpha & -3\gamma & 3 \cdot 4 & \dots & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ & & & & 3 \cdot 3 & \alpha & -(N-1)\gamma & N(N-1) \\ & & & & 0 & 3 \cdot 2 & \alpha & -N\gamma \\ 0 & & & \dots & 0 & 0 & 3 \cdot 1 & \alpha \end{pmatrix}$$

does not vanish. The first polynomials are given by

- $N=0$: $\beta = 3$, $\alpha = 0$ and $p_0(\rho) = 1$;
- $N=1$: $\beta = 6$, $\alpha^2 + 3\gamma = 0$ and $p_1(\rho) = \rho - \alpha/3$;
- $N=2$: $\beta = 9$, $\alpha^3 + 12\alpha\gamma + 36 = 0$ and

$$p_2(\rho) = \rho^2 - \frac{\alpha}{3}\rho + \frac{\alpha^2}{36} - \frac{1}{\alpha};$$

- $N=3$: $\beta = 12$, $\alpha^4 + 30\alpha^2\gamma + 216\alpha + 81\gamma^2 = 0$ and

$$p_3(\rho) = \rho^3 - \frac{\alpha}{3}\rho^2 + \left(\frac{\gamma}{2} + \frac{\alpha^2}{18}\right)\rho - \frac{\alpha^3}{162} - \frac{7}{54}\alpha\gamma - \frac{2}{3};$$

- $N=4$: $\beta = 15$, $\alpha^5 + 60\alpha^3\gamma + 756\alpha^2 + 576\alpha\gamma^2 + 5184\gamma = 0$ and

$$p_4(\rho) = \rho^4 - \frac{\alpha}{3}\rho^3 + \left(\frac{\alpha^2}{18} + \frac{2}{3}\gamma\right)\rho^2 - \left(\frac{\alpha^3}{162} + \frac{5}{27}\alpha\gamma + \frac{4}{3}\right)\rho + \frac{\alpha^4}{1944} + \frac{2}{81}\alpha^2\gamma + \frac{5}{18}\alpha + \frac{\gamma^2}{9}.$$

Note that the above list extends the one given in [16]. The special case $b_1 = 0 = b_2$ and $b_0 = 1$ and in particular the boundary value problem

$$-\frac{d^2v}{d\rho^2} + P(\rho)v = Ev, \quad v(-\infty) = v(+\infty) = 0, \quad P(\rho) = \frac{9}{4}\rho^4 - a_2\rho^2 - a_1\rho - a_0 \quad (26)$$

has been considered by [25]. Note that the boundary condition is equivalent to the requirement $v \in L^2(\mathbb{R})$. According to [26, 27] the spectrum of the above problem is discrete, all eigenvalues are real and simple and they can be arranged into an increasing sequence $E_0 < E_1 < \dots$. Moreover, for $n \rightarrow \infty$ the asymptotic behaviour of the eigenvalues is [25]

$$E_n \approx \left[\frac{4\pi n}{2B(3/2, 1/4)} \right]^{4/3} = \left[\frac{3\Gamma^2(3/4)}{\sqrt{2\pi}} n \right]^{4/3},$$

where B denotes the Euler function. Furthermore, all non-real zeros of the eigenfunctions of the problem (26) belong to the imaginary axis (see Theorem 1 in [28]).

We conclude this section by deriving a formula for the solution of the recurrence relation (25). By shifting the index according to the prescription $n \rightarrow n+1$ we can rewrite (25) as a third order homogeneous linear difference equation

$$w_{n+3} = -\pi_1(n, \gamma)w_{n+2} - \pi_2(n, \alpha)w_{n+1} - \pi_3(n, \beta)w_n \quad (27)$$

with $n \geq -2$ and

$$\pi_1(n, \gamma) = -\frac{\gamma}{n+3}, \quad \pi_2(n, \alpha) = \frac{\alpha}{(n+2)(n+3)}, \quad \pi_3(n, \beta) = \frac{\beta - 3(n+1)}{(n+2)(n+3)}, \quad w_{-2} = w_{-1} = 0.$$

By letting $n = 0$ we obtain w_3 in terms of w_2, w_1 , and w_0 , namely

$$w_3 = \frac{\gamma}{3}w_2 - \frac{\alpha}{2 \cdot 3}w_1 - \frac{\beta - 3}{2 \cdot 3}w_0.$$

Furthermore, for $n = -2$ and $n = -1$ we get, respectively

$$w_1 = \gamma w_0, \quad w_2 = \frac{1}{2}(\gamma^2 - \alpha)w_0$$

from which it follows that

$$w_3 = \frac{1}{3!} [\gamma^3 - 2\alpha\gamma - (\beta - 3)] w_0, \quad w_4 = \frac{1}{4!} [\gamma^4 - 3\alpha\gamma^2 + \alpha^2 - 3\gamma(\beta - 5)] w_0.$$

Hence, w_n can be evaluated once w_0 is assigned. Note also that solving (27) under the requirement $w_{-2} = w_{-1} = 0$ is equivalent to solve the same recurrence relation subject to the initial conditions

$$w_0 = 1, \quad w_1 = \gamma, \quad w_2 = \frac{\gamma^2 - \alpha}{2}. \quad (28)$$

Since our difference equation is linear and of third order, it will have a unique solution which can be expressed as a linear combination of three linear independent solutions $w_n^{(1)}$, $w_n^{(2)}$, and $w_n^{(3)}$. Once we find by other means these three particular solutions, we can check their linear independence by verifying that the Casoratian W_n of the solutions $w_n^{(1)}$, $w_n^{(2)}$, and $w_n^{(3)}$ defined by

$$W_n = \det \begin{pmatrix} w_n^{(1)} & w_n^{(2)} & w_n^{(3)} \\ w_{n+1}^{(1)} & w_{n+1}^{(2)} & w_{n+1}^{(3)} \\ w_{n+2}^{(1)} & w_{n+2}^{(2)} & w_{n+2}^{(3)} \end{pmatrix}$$

does not vanish for any value of n . On the other hand, Abel's lemma allows us to compute the Casoratian even without knowing explicitly the solutions $w_n^{(i)}$ with $i = 1, 2, 3$. Applying (2.2.10) in [29] we find

$$W_n = (-)^{3n} \prod_{k=0}^{n-1} \pi_3(k, \beta) W_0 = \frac{2(-)^{3n}(n+2)}{(n+2)!^2} (\beta-3)(\beta-6) \cdots (\beta-3n) W_0.$$

It is clear that we will have three linearly independent solutions provided that $\beta \neq 3(k+1)$ for all $k \in \mathbb{N}$ and $W_0 \neq 0$. If this is the case, then the general solution of (27) can be written as

$$w_n = C_1 w_n^{(1)} + C_2 w_n^{(2)} + C_3 w_n^{(3)}$$

with constants C_1 , C_2 , and C_3 determined by the initial condition (28). In order to solve the recurrence relation (27) we first rewrite it as a system of first order equations of dimension 3, namely

$$Z_{n+1} = A(\alpha, \beta, \gamma; n) Z_n \tag{29}$$

with

$$Z_n = \begin{pmatrix} Z_n^{(1)} \\ Z_n^{(2)} \\ Z_n^{(3)} \end{pmatrix} = \begin{pmatrix} w_n \\ w_{n+1} \\ w_{n+2} \end{pmatrix}, \quad A(\alpha, \beta, \gamma; n) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\pi_3(n, \beta) & -\pi_2(n, \alpha) & -\pi_1(n, \gamma) \end{pmatrix},$$

where A is the so-called companion matrix of the recurrence relation (27). Clearly, this matrix is non-singular if its determinant does not vanish and this condition translates into the requirement $\beta \neq 3(n+1)$ for any $n \in \mathbb{N}$. We want to solve (29) subject to the initial condition

$$Z_0 = \begin{pmatrix} w_1 \\ w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} 1 \\ \gamma \\ \frac{\gamma^2 - \alpha}{2} \end{pmatrix}.$$

By Theorem 3.4 in [29] this initial value problem has a unique solution U_n such that $U_0 = Z_0$. From (29) we have

$$U_1 = A(\alpha, \beta, \gamma; 0) Z_0, \quad U_2 = A(\alpha, \beta, \gamma; 1) Z_1 = A(\alpha, \beta, \gamma; 1) A(\alpha, \beta, \gamma; 0) Z_0$$

and by induction we conclude that

$$U_n = \left(\prod_{k=0}^{n-1} A(\alpha, \beta, \gamma; k) \right) Z_0,$$

where

$$\prod_{k=0}^{n-1} A(\alpha, \beta, \gamma; k) = \begin{cases} A(\alpha, \beta, \gamma; n-1) A(\alpha, \beta, \gamma; n-2) \cdots A(\alpha, \beta, \gamma; 0) & \text{if } n \geq 0 \\ \mathbb{I}_3 & \text{if } n = 0. \end{cases}$$

Even though the above formula contains the solution of the recurrence relation (27), it is difficult to extract from it the behaviour of the solution w_n as $n \rightarrow \infty$. The next result circumvents this problem by investigating the asymptotic behaviour of w_n with the help of the so-called Z-transform method which reduces the study of a linear difference equation to an examination of a corresponding complex function.

Theorem IV.1 *For the recurrence relation (27) with the initial condition (28) it holds*

$$\lim_{n \rightarrow \infty} w_n = 0.$$

Proof. Applying the definition of the Z-transform to our recurrence relation (27), i.e.

$$\tilde{x}(z) = Z(w_n) = \sum_{j=0}^{\infty} \frac{w_j}{z^j}, \quad z \in \mathbb{C} \setminus \{0\},$$

we find with the help of the properties of the Z-transform listed in Ch. 6 in [29] that the original recurrence relation gives rise to the following second order, linear, non homogeneous complex differential equation

$$z^5 \frac{d^2 \tilde{x}}{dz^2} + z(2z^3 + \gamma z^2 + 3) \frac{d\tilde{x}}{dz} + (\alpha z + \beta - 3)\tilde{x} = (2w_2 - \gamma w_1 + \alpha w_0)z. \quad (30)$$

The initial condition (28) implies that $2w_2 - \gamma w_1 + \alpha w_0 = 0$ and therefore, we are left with the following homogeneous differential equation

$$z^5 \frac{d^2 \tilde{x}}{dz^2} + z(2z^3 + \gamma z^2 + 3) \frac{d\tilde{x}}{dz} + (\alpha z + \beta - 3)\tilde{x} = 0. \quad (31)$$

By means of the substitution $\omega = 1/z$ the above equation can be transformed into the triconfluent Heun equation (20). Hence, a particular solution is given by $\tilde{x}_1(\omega) = T_1(\alpha, \beta, \gamma; \omega)$ with T_1 defined as in (23). In order to construct a linearly independent set we make use of Proposition 4.2 with $j = 1$ in [30]. This gives

$$\gamma T_2(\alpha, \beta, \gamma; \omega) + T_1(\alpha, \beta, \gamma; \omega) = e^{\omega^3 + \gamma \omega} T_1(\alpha, -\beta, \gamma; -\omega)$$

with T_2 defined as in (23). Hence, the general solution of (31) for $\gamma \neq 0$ is

$$\tilde{x}(z) = C_1 T_1(\alpha, \beta, \gamma; 1/z) + C_2 e^{\frac{\gamma z^2 + 1}{z^3}} T_1(\alpha, -\beta, \gamma; -1/z),$$

where the corresponding series defining T_1 converges uniformly provided that $z \neq 0$. Then, the Final Value Theorem for the Z-transform [29] implies that

$$\lim_{n \rightarrow \infty} w_n = \lim_{z \rightarrow 1} (z - 1) \tilde{x}(z) = 0.$$

If $\gamma = 0$, we can still use the particular solution \tilde{x}_1 found before together with Abel's formula. Let \tilde{x}_2 denote another particular solution of the triconfluent Heun equation arising from (30) after the transformation $\omega = 1/z$. Then, the Wronskian is given by $W(T_1(\alpha, \beta, 0; \omega); \tilde{x}_2(\omega)) = C \exp(\omega^3)$ for some $C \neq 0$. Employing the definition of the Wronskian we end up with the following first order, linear differential equation for \tilde{x}_2 , namely

$$\frac{d\tilde{x}_2}{d\omega} - \frac{T_1'(\alpha, \beta, 0; \omega)}{T_1(\alpha, \beta, 0; \omega)} \tilde{x}_2 = C \frac{e^{\omega^3}}{T_1(\alpha, \beta, 0; \omega)}.$$

The corresponding integrating factor is given by $\mu(\omega) = T_1^{-1}(\alpha, \beta, 0; \omega)$ and hence

$$\tilde{x}_2(\omega) = T_1(\alpha, \beta, 0; \omega) \int \frac{e^{\omega^3}}{T_1^2(\alpha, \beta, 0; \omega)} d\omega.$$

Finally, the general solution of (31) for $\gamma = 0$ is represented by

$$\tilde{x}(z) = T_1(\alpha, \beta, 0; 1/z) \left(D_1 + D_2 \int \frac{e^{1/z^3}}{z^2 T_1^2(\alpha, \beta, 0; 1/z)} dz \right).$$

Taking into account that the following expansion holds for $|z - 1| < 1$

$$\int \frac{e^{1/z^3}}{z^2 T_1^2(\alpha, \beta, 0; 1/z)} dz = \frac{e(z-1)}{T_1^2(\alpha, \beta, 0; 1)} + \mathcal{O}(z-1)^2,$$

once again the Final Value Theorem implies that $\lim_{z \rightarrow 1} (z - 1) \tilde{x}(z) = 0$ and this completes the proof. \square

We conclude the analysis of the recurrence relation (27) by constructing asymptotic expansions valid as $n \rightarrow \infty$ consisting of an exponential leading term multiplied by a descending series. These kind of expansions are called Birkhoff series [31, 32]. To construct such series we will assume as in [33] that

$$w_n = \mathcal{E}_n K_n, \quad \mathcal{E}_n = e^{\mu_0 n \ln n + \mu_1 n^\theta}, \quad K_n = e^{\alpha_1 n^{\tilde{\beta}} + \alpha_2 n^{\tilde{\beta}-1/\tilde{\rho}} + \dots} \quad (32)$$

with $\alpha_1 \neq 0$, $\tilde{\beta} = j/\tilde{\rho}$, $0 \leq j < \rho$, $\rho \geq 1$ and $\mu_0 \rho$ taking integer values. Then, for $k = 1, 2, 3$ we have

$$\frac{w_{n+k}}{w_n} = n^{\mu_0 k} \lambda^k \left[1 + \frac{A}{n} + \frac{B}{n^2} + \frac{C}{n^3} + \dots \right] e^{\alpha_1 \tilde{\beta} k n^{\tilde{\beta}-1} + \alpha_2 (\tilde{\beta} - \frac{1}{\tilde{\rho}}) k n^{\tilde{\beta} - (1/\tilde{\rho}) - 1} + \dots}, \quad \lambda = e^{\mu_0 + \mu_1}. \quad (33)$$

where

$$\begin{aligned} A &= k\theta + \frac{\mu_0 k^2}{2}, \\ B &= \frac{\mu_0^2 k^4}{8} - \frac{\mu_0 k^3}{6} + \frac{1}{2}\mu_0 k^3 \theta + \frac{k^2}{2}\theta(\theta - 1), \\ C &= \frac{\mu_0^3 k^6}{48} - \frac{\mu_0^2 k^5}{12} + \frac{\mu_0 k^4}{12} + \theta k \left(\frac{\mu_0^2 k^4}{8} - \frac{\mu_0 k^3}{6} \right) + \frac{\mu_0 k^4}{4}\theta(\theta - 1) + \frac{k^3}{6}\theta(\theta - 1)(\theta - 2). \end{aligned}$$

To construct Birkhoff series for the recurrence relation (27) it turns out to be convenient to rewrite it as

$$(\beta - 3 - 3n)w_n + \alpha w_{n+1} - \gamma(n+2)w_{n+2} + (n+2)(n+3)w_{n+3} = 0.$$

Substituting (32) and (33) into the above equation, we find the equation to be formally satisfied is

$$\begin{aligned} &\beta - 3 - 3n + \alpha \lambda n^{\mu_0} \left[1 + \frac{\theta + \mu_0/2}{n} + \mathcal{O}(n^{-2}) \right] e^{\alpha_1 \tilde{\beta} n^{\tilde{\beta}-1} + \alpha_2 (\tilde{\beta} - \frac{1}{\tilde{\rho}}) n^{\tilde{\beta} - (1/\tilde{\rho}) - 1} + \dots} \\ &- \gamma \lambda^2 n^{2\mu_0+1} \left[1 + \frac{2(\theta + \mu_0 + 1)}{n} + \mathcal{O}(n^{-2}) \right] e^{2\alpha_1 \tilde{\beta} n^{\tilde{\beta}-1} + 2\alpha_2 (\tilde{\beta} - \frac{1}{\tilde{\rho}}) n^{\tilde{\beta} - (1/\tilde{\rho}) - 1} + \dots} \\ &+ \lambda^3 n^{3\mu_0+2} \left[1 + \frac{3\theta + \frac{9}{2}\mu_0 + 5}{n} + \mathcal{O}(n^{-2}) \right] e^{3\alpha_1 \tilde{\beta} n^{\tilde{\beta}-1} + 3\alpha_2 (\tilde{\beta} - \frac{1}{\tilde{\rho}}) n^{\tilde{\beta} - (1/\tilde{\rho}) - 1} + \dots} = 0. \end{aligned}$$

Obviously, it must be $\mu_0 = -1/3$ and this implies $\lambda = 3^{1/3}$. This further requires that $\tilde{\rho} = 3$ and $\tilde{\beta} = 1/3$. Expanding the exponentials gives

$$(3\alpha_1 - 3^{2/3}\gamma)n^{1/3} + \beta - 3 + 3 \left(3\theta + \frac{7}{2} \right) + (\text{lower order terms}) = 0.$$

Thus, $\alpha_1 = \gamma/3^{1/3}$ and $\theta = -5/6 - \beta/9$. Hence, the first asymptotic series is

$$w_n^{(1)} \sim \left(\frac{3e}{n} \right)^{n/3} n^{-\frac{5}{6} - \frac{\beta}{9}} e^{\gamma \sqrt[3]{\frac{n}{3}}} \left[1 + \frac{c_1}{n^{1/3}} + \frac{c_2}{n^{2/3}} + \frac{c_3}{n} + \mathcal{O}(n^{-4/3}) \right]$$

with

$$\begin{aligned} c_1 &= \frac{1}{\sqrt[3]{9}} \left(\alpha - \frac{\gamma^2}{6} \right), \\ c_2 &= \frac{1}{2\sqrt[3]{9}} \left[\gamma \left(1 - \frac{\beta}{3} \right) + \left(\alpha - \frac{\gamma^2}{6} \right)^2 \right], \\ c_3 &= \frac{1}{324} \left[-108c_1^3 + 324c_1c_2 + (8\sqrt[3]{1089} - 36\sqrt[3]{9})\gamma c_1 + 6\beta^2 + 18\beta - 2\gamma^3 - 81 \right]. \end{aligned}$$

According to [33] other two formal series solutions can be obtained by letting $n \rightarrow ne^{2\pi ki}$ with $k = 0, 1, \dots, \tilde{\rho} - 1$. In the present case we have $\tilde{\rho} = 3$ and the corresponding Birkhoff series are thus

$$\begin{aligned} w_n^{(2)} &\sim \left(\frac{3e}{n} \right)^{n/3} n^{-\frac{5}{6} - \frac{\beta}{9}} e^{-\frac{2}{3}\pi i n - \frac{1}{2}(1-i\sqrt{3})\gamma \sqrt[3]{\frac{n}{3}}} \left[1 - \frac{(1+i\sqrt{3})c_1}{2n^{1/3}} - \frac{(1-i\sqrt{3})c_2}{2n^{2/3}} + \frac{c_3}{n} + \mathcal{O}(n^{-4/3}) \right], \\ w_n^{(3)} &\sim \left(\frac{3e}{n} \right)^{n/3} n^{-\frac{5}{6} - \frac{\beta}{9}} e^{-\frac{4}{3}\pi i n - \frac{1}{2}(1+i\sqrt{3})\gamma \sqrt[3]{\frac{n}{3}}} \left[1 - \frac{(1-i\sqrt{3})c_1}{2n^{1/3}} - \frac{(1+i\sqrt{3})c_2}{2n^{2/3}} + \frac{c_3}{n} + \mathcal{O}(n^{-4/3}) \right]. \end{aligned}$$

Last but not least, note that the above asymptotic results for $w_n^{(i)}$ with $i = 1, 2, 3$ tend to zero for $n \rightarrow \infty$ as we would expect from Theorem IV.1.

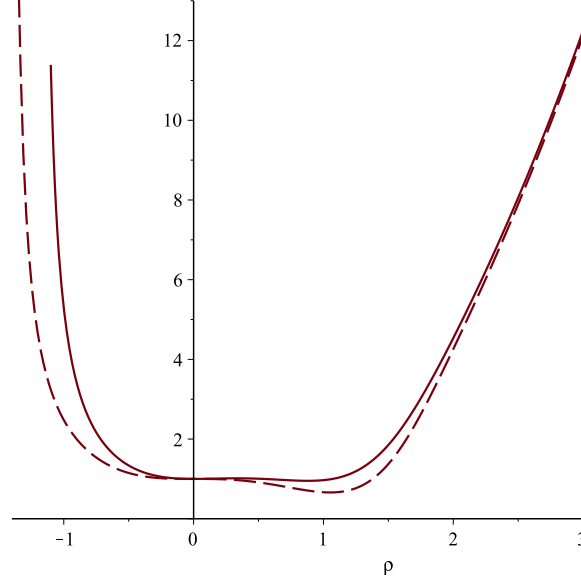


FIG. 4: Behaviour of the superpotential W for different values of the parameters: solid line corresponds to $\alpha = 0 = c_1$, $\beta = -1$, $\gamma = -2$; dash line $\alpha = 0 = c_1 = \beta$, $\gamma = -2$.

V. ANALYSIS OF THE TRICONFLUENT HEUN EQUATION USING SUPERSYMMETRIC QUANTUM MECHANICS

For the following analysis it is convenient to rewrite the triconfluent Heun equation (5) as $(L_T v)(\rho) = 0$ where

$$L_T = -\frac{d^2}{d\rho^2} + \Omega(\rho), \quad \Omega(\rho) = -A_0 - A_1\rho - A_2\rho^2 + \frac{9}{4}\rho^4.$$

We look for a factorization of the operator L_T having the form

$$L_T = AA^\dagger = \left(\frac{d}{d\rho} + W(\rho) \right) \left(-\frac{d}{d\rho} + W(\rho) \right).$$

This will be the case if the unknown function W satisfies the Riccati equation

$$\frac{dW}{d\rho} + W^2(\rho) = \Omega(\rho).$$

We solved the above equation with the help of the software package Maple 18 and we found that the superpotential W is given by

$$W(\rho) = \frac{1}{6} \frac{e^{f(\rho)} F(\alpha, \beta, \gamma; \rho) - c_1 e^{-f(\rho)} F(\alpha, -\beta, \gamma; -\rho)}{e^{f(\rho)} T_1(\alpha, \beta, \gamma; \rho) + c_1 e^{-f(\rho)} T_1(\alpha, -\beta, \gamma; -\rho)}$$

where c_1 is an arbitrary integration constant and

$$f(\rho) = -\frac{\gamma}{2}\rho - \frac{\rho^3}{2}, \quad F(\alpha, \beta, \gamma; \rho) = -3(\gamma + 3\rho^2)T_1(\alpha, \beta, \gamma; \rho) + 6\dot{T}_1(\alpha, \beta, \gamma; \rho).$$

with α , β , and γ as in (21). Note that as a byproduct result we also obtained a new factorization of the triconfluent Heun equation since it does not coincide with the one offered in [16].

Combining the operators A and A^\dagger we can construct Hamiltonians

$$\mathcal{H}_- = A^\dagger A = -\frac{d^2}{d\rho^2} + V_-(\rho), \quad \mathcal{H}_+ = AA^\dagger = L_T = -\frac{d^2}{d\rho^2} + V_+(\rho)$$

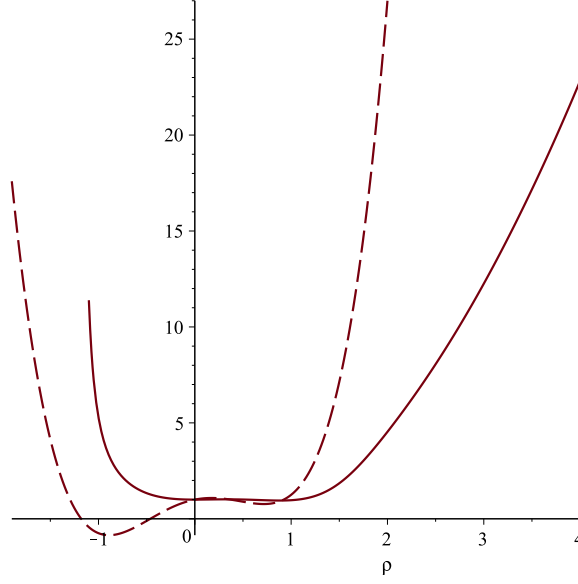


FIG. 5: Behaviour of the partner potentials V_- (solid line) and V_+ (dash line) for $\alpha = 0$, $\beta = -1$, $\gamma = -2$, $c_1 = 0$.

where V_+ and V_- are the supersymmetric partner potentials and they are given by

$$V_{\pm}(\rho) = W^2(\rho) \pm \frac{dW}{d\rho}.$$

Clearly, $V_+(\rho) = \Omega(\rho)$ and moreover the other partner potential can be expressed as

$$V_-(\rho) = 2W^2(\rho) - \Omega(\rho) = \Omega(\rho) - 2\frac{dW}{d\rho}.$$

If we denote the eigenstates of the Hamiltonians \mathcal{H}_- and \mathcal{H}_+ by $v_n^{(-)}$ and $v_n^{(+)}$, respectively, then $v_n^{(\pm)}$ must satisfy the eigenvalue equations

$$\mathcal{H}_- v_n^{(-)} = E_n^{(-)} v_n^{(-)}, \quad \mathcal{H}_+ v_n^{(+)} = E_n^{(+)} v_n^{(+)}.$$

In order to establish whether or not the supersymmetry is unbroken we need to find out if the ground state of \mathcal{H}_- has zero energy, that is $E_0^{(-)} = 0$.

VI. ANALYSIS OF THE ZERO ENERGY STATE

Let ψ_0 denote the zero energy solution, i.e. $E = 0$, associated to an Hamiltonian H_0 such that $H_0\psi_0 = 0$ and $H_0 = d^2/dr^2 + V_{eff}(r)$ where the effective potential has been already computed in Section II. For the potential V_3 with $e_0 = e_1 = e_2 = 0$ corresponding to the choice $a_0 = a_1 = a_2 = 0$ we find that the zero energy solution can be written as a linear combination of Bessel functions as follows

$$\psi_0(r) = \sqrt{r} \left[C_1 J_{\sqrt{7}/6} \left(\frac{2\sqrt{e_4}}{3} r^{3/2} \right) + C_2 Y_{\sqrt{7}/6} \left(\frac{2\sqrt{e_4}}{3} r^{3/2} \right) \right]$$

with integration constants C_1 and C_2 . This solution is not square integrable since it oscillates asymptotically as a plane wave. Therefore, it is obvious that we are handling here the scattering solution for a spacial case of the energy. Note that from the definition of e_4 we can conclude that it is always strictly positive. Another case that can be solved exactly is the following. Consider the potential V_4 and let $a_0 = a_2 = b_0 = 0$. Then, the coefficients v_1 , v_3 , and v_4 vanishes whereas $v_0 = -a_1/b_1$. Hence, we have

$$V_4(r) = -\frac{5}{36r^2} + v_5 r^2 + v_0$$

with $v_5 > 0$. For the above potential the Schrödinger equation

$$\frac{d^2\psi}{dr^2} + V_4(r)\psi(r) = E\psi(r)$$

can be exactly solved in terms of a linear combination of Whittaker functions given by

$$\psi_E(r) = \frac{1}{\sqrt{r}} [C_1 M_{\alpha,\beta}(i\sqrt{v_5}r^2) + C_2 W_{\alpha,\beta}(i\sqrt{v_5}r^2)]$$

with

$$\alpha = \frac{i(E - v_0)}{4\sqrt{5}}, \quad \beta = \frac{\sqrt{14}}{12}.$$

Observe that if $E = v_0$ or $E = 0$ and $v_0 = 0$ we find that

$$\psi_{E=v_0}(r) = \sqrt{r} \left[C_1 J_\beta \left(\frac{\sqrt{v_5}}{2} r^2 \right) + C_2 Y_\beta \left(\frac{\sqrt{v_5}}{2} r^2 \right) \right].$$

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- [1] S. Flügge, *Practical Quantum Mechanics*, Springer-Verlag: Berlin, Heidelberg, New York, 1999
 - [2] J. Dereziński and M. Wrochna, “Exactly Solvable Schrödinger Operators”, *Ann. Henri Poincaré* **12** (2011) 397 and references therein
 - [3] A. Lamieux and A. K. Bose, “Construction de potentiels pour lesquels l’équation de Schrödinger est soluble”, *Annales de l’Inst. H. Poincaré* **X** (1969) 259
 - [4] G. A. Natanson, “Study of the one-dimensional Schrödinger equation generated from the hypergeometric equation”, *Vestn. Leningr. Univ.* **10** (1971) 22
 - [5] G. Natanson, “Heun-Polynomial Representation of Regular-at-Infinity Solutions for the Basic SUSY Ladder of Hyperbolic Pöschl-Teller Potentials Starting from the Reflectionless Symmetric Potential Well, arXiv:1410.1515 (2014)
 - [6] D. Batic, R. Williams and M. Nowakowski, “Potentials of the Heun class”, *J. Phys. A: Math. Theor.* **46** (2013) 245204
 - [7] R. Milson, “Liouville Transformation and Exactly Solvable Schrödinger Equations”, *Int. J. Theor. Phys.* **37** (1998) 1735
 - [8] A. Messiah, *Quantum Mechanics*, North Holland Publishing Company, 1967
 - [9] S. C. Zhang, T. H. Hansen and S. Kivelson, “Effective-Field-Theory Model for the Fractional Quantum Hall Effect”, *Phys. Rev. Lett.* **62** (1989) 82
 - [10] A. Fetter, C. Hanna and R. Laughlin, “Random-phase approximation in the fractional-statistics gas”, *Phys. Rev. B* **39** (1989) 9679
 - [11] P. B. Wiegmann, “Superconductivity in strongly correlated electronic systems and confinement versus deconfinement phenomenon”, *Phys. Rev. Lett.* **60** (1988) 821
 - [12] A. M. Polyakov, “Fermi-Bose transmutations induced by gauge fields”, *Mod. Phys. Lett. A* **3** (1988) 325
 - [13] A. Lerda, *Anyons: Quantum Mechanics of Particles with Fractional Statistics*, Springer Verlag, 1992
 - [14] C. Manuel and R. Tarrach, “Contact Interactions of Anyons”, *Phys. Lett. B* **268** (1991) 222
 - [15] B. Roy, A. O. Barut and P. Roy, “On the dynamical group of the system of two anyons with Coulomb interaction”, *Phys. Lett. A* **172** (1993) 316
 - [16] A. Ronveaux, *Heun’s Differential Equations*, Oxford University Press, 1995
 - [17] F. Hioe, D. MacMillen and E. Montroll, “Quantum theory of anharmonic oscillators. II. Energy levels of oscillators with $x^{2\alpha}$ anharmonicity”, *J. Math. Phys.* **17** (1976) 1320
 - [18] J. Q. Liang and H. J. W. Müller-Kirsten, “Anharmonic Oscillator Equations: Treatment Parallel to Mathieu Equation”, *Transactions of the IRE Professional Group* (2004), arXiv:quant-ph/0407235
 - [19] A. Schulze-Halberg, “Quasi-Exactly Solvable Singular Fractional Power Potentials Emerging from the Triconfluent Heun Equation”, *Phys. Scr.* **65** (2002) 373
 - [20] K. Bay, W. Lay and A. Akopyan, “Avoided Crossings of the quartic oscillator”, *J. Phys. A: Math. Gen.* **30** (1997) 3057
 - [21] A. Ushveridze, *Quasi-exactly solvable models in quantum mechanics*, Inst. of Physics Publ., Bristol, 1994
 - [22] A. K. Bose, “A Class of Solvable Potentials”, *Nuovo Cim.* **32** (1964) 679
 - [23] I. S. Gradshteyn and I. M. Ryzbik, *Table of Integrals, Series, and Products*, Academic Press, 2007
 - [24] L. Yang, “Recent Advances on Determining the Number of Real Roots of Parametric Polynomials”, *J. Symb. Comp.* **28** (1998) 225
 - [25] A. Eremenko, A. Gabrielov and B. Shapiro, High-energy eigenfunctions of one-dimensional Schrödinger operators with polynomial potentials, *Comp. Meth. Function Theory* **8** (2008) 513
 - [26] F. A. Berezin and M. A. Shubin, *The Schrödinger equation*, Kluwer, Dordrecht, 1991
 - [27] Y. Sibuya, *Global theory of a second order linear ordinary differential equation with a polynomial coefficient*, North-Holland Publishing Co., Amsterdam-Oxford, 1975

- [28] A. Eremenko, A. Gabrielov and B. Shapiro, Zeroes of eigenfunctions of some anharmonic oscillators, *Ann. Inst. Fourier, Grenoble*, **58** (2008) 603
- [29] S. Elaydi, *An Introduction to Difference Equations*, Springer Verlag, 2000
- [30] A. Decarreau, P. Maroni and A. Robert, “Sur les equations confluentes de l’equation de Heun”, *Ann. Soc. Sci. Brux.* **92** (1978) 151
- [31] G. D. Birkhoff, “ Formal theory of irregular linear difference equations”, *Acta Math.* **54** (1930) 205
- [32] G. D. Birkhoff and W. J. Trjitzinsky, “Analytic theory of singular difference equations”, *Acta Math.* **60** (1932) 1
- [33] J. Wimp, *Computation with Recurrence Relations*, Pitman Press, 1984